

Locally Inertial Reference Frames in Lorentzian and Riemann-Cartan Spacetimes

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Abstract

In this paper we scrutinize the concept of locally inertial reference frames (**LIRF**) in Lorentzian and Riemann-Cartan spacetime structures. We present rigorous mathematical definitions for those objects, something that needs preliminary a clear mathematical distinction between the concepts of observers, reference frames, naturally adapted coordinate functions to a given reference frame and which properties may characterize an inertial reference frame (if any) in the Lorentzian and Riemann-Cartan structures. We hope to have clarified some eventual obscure issues associated to the concept of **LIRF** appearing in the literature, in particular the relationship between **LIRF**s in Lorentzian and Riemann-Cartan spacetimes and Einstein's most happy thought, i.e., the equivalence principle.

1 Introduction

In this note we investigate if it is possible to have in a general Riemann-Cartan spacetime a locally inertial reference frame in an analogous sense in which this concept is defined in a Lorentzian spacetime that models possible gravitational fields in General Relativity.

To answer the above question which is affirmative in a well defined sense we are going to recall the precise definitions of the following fundamental concepts:

- (i) definition of a *general reference frame* in Lorentzian and Riemann-Cartan spacetimes;
- (ii) definition of *observers* in a Lorentzian or Riemannian spacetime;
- (ii) classification of *reference frames* in Lorentzian spacetimes¹;

¹The classification of reference frames will not be presented in this paper. For the Lorentzian spacetime case, see [18].

(iii) definition of an *inertial reference frame* (**IRF**) in Minkowski spacetime;
(iv) definition of a *locally inertial reference frame* (**LIRF**) in Lorentzian and Riemann-Cartan spacetimes.

However, to be possible to present precise definitions of the concepts just mentioned we need to recall some basic facts of differential geometry and fix some notation. This will be done in Section 2.

2 Lorentzian and Riemann-Cartan Spacetimes

To start we introduce a Lorentzian manifold as a pair $\langle M, \mathbf{g} \rangle$ where M is a 4-dimensional manifold and $\mathbf{g} \in \sec T_0^2 M$ is a Lorentz metric of signature $(1, 3)$. We suppose that $\langle M, \mathbf{g} \rangle$ is orientable by a global volume form $\tau_{\mathbf{g}} \in \sec \bigwedge^4 T^* M$ and also time orientable by an equivalence relation here denoted \uparrow . We next introduce on M two metric compatible connections, namely \mathring{D} the Levi-Civita connection of \mathbf{g} and D a general Riemann-Cartan connection.

Definition 1 We call the pentuple $\langle M, \mathbf{g}, \mathring{D}, \tau_{\mathbf{g}}, \uparrow \rangle$ a Lorentzian spacetime and the pentuple $\langle M, \mathbf{g}, D, \tau_{\mathbf{g}}, \uparrow \rangle$ a Riemann-Cartan spacetime.

Remark 2 Minkowski spacetime structure is denoted by $\langle M \simeq \mathbb{R}^4, \eta, \overset{m}{D}, \tau_{\eta}, \uparrow \rangle$.

Let $U, V, W \subset M$ with $U \cap V \cap W \neq \emptyset$ and introduce the local charts (φ, U) and (ψ, V) and (χ, W) with coordinate functions $\langle \xi^\mu \rangle, \langle x^\mu \rangle, \langle x'^\mu \rangle$ respectively. Recall to fix notation that, e.g., given $p \in M$ and $\mathcal{V} \subset \mathbb{R}^4$ we have

$$\psi : V \rightarrow \mathcal{V}, \quad \psi(p) = (x^0(p), x^1(p), x^2(p), x^3(p)). \quad (1)$$

The coordinate chart ψ determines a so-called coordinate basis for TV denoted by $\langle e_\mu = \partial/\partial x^\mu \rangle$. We denoted by $\langle \vartheta^\mu = dx^\mu \rangle$ a basis for T^*V dual to $\langle e_\mu = \partial/\partial x^\mu \rangle$, this meaning that $\vartheta^\mu(e_\nu) = \delta_\nu^\mu$.

We also write

$$\begin{aligned} \mathbf{g} &= g_{\mu\nu} \vartheta^\mu \otimes \vartheta^\nu = g^{\mu\nu} \vartheta_\mu \otimes \vartheta_\nu, \\ g_{\mu\nu} &:= \mathbf{g}(e_\mu, e_\nu), \quad g^{\mu\nu} := \mathbf{g}(\vartheta^\mu, \vartheta^\nu), \end{aligned} \quad (2)$$

where we denoted by $\langle e^\mu \rangle$ the reciprocal basis of $\langle e_\mu \rangle$, i.e., we have

$$\mathbf{g}(e^\mu, e_\nu) = \delta_\nu^\mu. \quad (3)$$

Moreover, we denote by $g \in \sec T_2^0 M$ the metric on the cotangent bundle and write

$$g = g^{\mu\nu} e_\mu \otimes e_\nu = g_{\mu\nu} e^\mu \otimes e^\nu. \quad (4)$$

Of course, $g^{\mu\alpha} g_{\alpha\nu} = \delta_\nu^\mu$. Also, we denoted by $\langle \vartheta_\mu \rangle$ the reciprocal basis of $\langle \vartheta^\mu \rangle$, i.e., $g(\vartheta^\mu, \vartheta_\nu) = \delta_\nu^\mu$.

A curve in M is a mapping

$$c : \mathbb{R} \supset I \rightarrow M, \quad \tau \mapsto c(\tau).$$

As usual the *tangent vector field* to the curve c is denoted by c_* or by $\frac{d}{d\tau}$ as more convenient. In the coordinate basis $\langle e_\mu = \partial/\partial x^\mu \rangle$ we write

$$c_* = \frac{d}{d\tau} = v^\mu(\tau) \partial/\partial x^\mu|_{\gamma(\tau)} \quad (5)$$

In particular we write when $c(0) = p_o$,

$$c_*|_{\tau=0} = v^\mu e_\mu|_{p_o} \in T_{p_o}M. \quad (6)$$

To understand the reason for that notation, first take into account that the coordinate representation of c are the set of functions $x^\mu \circ c(\tau)$ that we denoted using a sloop notation simply by $x^\mu(\tau)$.

Now, consider a function $\mathbf{f} : V \rightarrow \mathbb{R}$ and denote by $f = \mathbf{f} \circ \psi^{-1} : \mathcal{V} \rightarrow \mathbb{R}$ its representation as functions of the coordinates $\langle x^\mu \rangle$. Moreover, consider the composite function $\mathbf{f} \circ c$ and its representative

$$f(x^\mu(\tau)) \quad (7)$$

Then the value of the function $\frac{d}{d\tau} \mathbf{f} \circ c$ at $c(\tau_o) = p_o$ is

$$\left. \frac{d}{d\tau} \mathbf{f} \circ c(\tau) \right|_{p_o} := \left. \frac{d}{d\tau} f(x^\mu(\tau)) \right|_{\tau=0} = v^\mu \partial f / \partial x^\mu|_{p_o}, \quad (8)$$

with

$$v^\mu := \left. \frac{dx^\mu(\tau)}{d\tau} \right|_{\tau=0}. \quad (9)$$

The metric structure permit to classify curves as *timelike*, *spacelike* and *lightlike*. We have

$$\begin{cases} \mathbf{g}(c_*, c_*) > 0 & \forall \tau \in I & c \text{ is timelike} \\ \mathbf{g}(c_*, c_*) < 0 & \forall \tau \in I & c \text{ is spacelike} \\ \mathbf{g}(c_*, c_*) = 0 & \forall \tau \in I & c \text{ is lighthlike} \end{cases} \quad (10)$$

For timelike curve $c : \tau \mapsto c(\tau)$, such that $\mathbf{g}(c_*, c_*) = 1$ the parameter τ is called the *proptime*.

Given $U, V \subset M$ and coordinate functions $\langle \xi^\mu \rangle, \langle x^\mu \rangle$ covering U and V for the structure $\langle M, \mathbf{g}, \hat{D}, \tau_{\mathbf{g}}, \uparrow \rangle$ and coordinate functions $\langle \zeta^\mu \rangle, \langle x^\mu \rangle$ covering U and V for the structure $\langle M, \mathbf{g}, D, \tau_{\mathbf{g}}, \uparrow \rangle$ we fix here the following notation

$$\begin{aligned} D_{e_\mu} \vartheta^\nu &:= -\Gamma_{\cdot\mu\alpha}^{\nu\cdot\cdot} \vartheta^\alpha, & D_{e_\mu} e_\nu &:= \Gamma_{\cdot\mu\nu}^{\alpha\cdot\cdot} e_\alpha, \\ D_{\partial/\partial \zeta^\mu} d\zeta^\nu &:= -\mathring{\Gamma}_{\cdot\mu\alpha}^{\nu\cdot\cdot} d\zeta^\alpha, & D_{\partial/\partial \zeta^\mu} \partial/\partial \zeta^\nu &:= \mathring{\Gamma}_{\cdot\mu\nu}^{\alpha\cdot\cdot} \partial/\partial \zeta^\alpha, \\ \hat{D}_{e_\mu} \vartheta^\nu &:= -\hat{\Gamma}_{\cdot\mu\alpha}^{\nu\cdot\cdot} \vartheta^\alpha, & \hat{D}_{e_\mu} e_\nu &:= \hat{\Gamma}_{\cdot\mu\nu}^{\alpha\cdot\cdot} e_\alpha, \\ \hat{D}_{\partial/\partial \xi^\mu} d\xi^\nu &:= -\hat{\Gamma}_{\cdot\mu\alpha}^{\nu\cdot\cdot} d\xi^\alpha, & \hat{D}_{\partial/\partial \xi^\mu} \partial/\partial \xi^\nu &= \hat{\Gamma}_{\cdot\mu\nu}^{\alpha\cdot\cdot} \partial/\partial \xi^\alpha. \end{aligned} \quad (11)$$

For the connection coefficients in coordinate basis $\langle \partial/\partial x'^\mu \rangle, \langle dx'^\mu \rangle$ we use $\Gamma_{\cdot\mu\alpha}^{\nu\cdot}$. Finally for an arbitrary basis $\langle e_\mu \rangle$ for $T(U \cap V \cap W)$ and dual basis $\langle \theta^\mu \rangle$ for $T^*(U \cap V \cap W)$ such that

$$[e_\mu, e_\nu] = c_{\cdot\mu\nu}^{\alpha\cdot} e_\alpha \quad (12)$$

we write

$$\begin{aligned} D_{e_\mu} \theta^\nu &:= -\gamma_{\cdot\mu\alpha}^{\nu\cdot} \theta^\alpha, & D_{e_\mu} e_\nu &:= \gamma_{\cdot\mu\nu}^{\alpha\cdot} e_\alpha, \\ \mathring{D}_{e_\mu} \theta^\nu &:= -\mathring{\gamma}_{\cdot\mu\alpha}^{\nu\cdot} \theta^\alpha, & \mathring{D}_{e_\mu} e_\nu &:= \mathring{\gamma}_{\cdot\mu\nu}^{\alpha\cdot} e_\alpha. \end{aligned} \quad (13)$$

2.1 Relation between $\Gamma_{\cdot\mu\nu}^{\lambda\cdot}$ and $\mathring{\Gamma}_{\cdot\mu\nu}^{\lambda\cdot}$

We have that²:

$$\Gamma_{\cdot\mu\nu}^{\lambda\cdot} = \mathring{\Gamma}_{\cdot\mu\nu}^{\lambda\cdot} + K_{\cdot\mu\nu}^{\lambda\cdot} \quad (14)$$

where

$$\begin{aligned} K_{\cdot\mu\nu}^{\lambda\cdot} &:= \frac{1}{2}(T_{\cdot\mu\nu}^{\lambda\cdot} + S_{\cdot\mu\nu}^{\lambda\cdot}) \\ &= \frac{1}{2}g^{\lambda\beta}g_{\beta\alpha}T_{\cdot\mu\nu}^{\alpha\cdot} - \frac{1}{2}g^{\lambda\sigma}g_{\mu\alpha}T_{\cdot\nu\sigma}^{\alpha\cdot} - \frac{1}{2}g^{\lambda\sigma}g_{\nu\alpha}T_{\cdot\mu\sigma}^{\alpha\cdot} \\ &= \frac{1}{2}(T_{\cdot\mu\nu}^{\lambda\cdot} - T_{\mu\nu}^{\cdot\lambda} + T_{\nu\cdot\mu}^{\lambda\cdot}). \end{aligned} \quad (15)$$

and

$$T_{\cdot\mu\nu}^{\lambda\cdot} = \Gamma_{\cdot\mu\nu}^{\lambda\cdot} - \Gamma_{\cdot\nu\mu}^{\lambda\cdot} = -T_{\cdot\nu\mu}^{\lambda\cdot}, \quad (16)$$

$$S_{\cdot\mu\nu}^{\lambda\cdot} = -g^{\lambda\sigma}(g_{\mu\alpha}T_{\cdot\nu\sigma}^{\alpha\cdot} + g_{\nu\alpha}T_{\cdot\mu\sigma}^{\alpha\cdot}) = S_{\cdot\nu\mu}^{\lambda\cdot}. \quad (17)$$

2.2 Torsion and Curvature Tensors

Definition 3 Let $\mathbf{u}, \mathbf{v} \in \sec TM$. The torsion and curvature operations of a connection ∇ are respectively the mappings: $\tau : \sec(TM \otimes TM) \rightarrow \sec TM$ and $\rho : \sec(TM \otimes TM) \rightarrow \text{End}(\sec TM)$ given by

$$\tau(\mathbf{u}, \mathbf{v}) = \nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u} - [\mathbf{u}, \mathbf{v}], \quad (18)$$

$$\rho(\mathbf{u}, \mathbf{v}) = \nabla_{\mathbf{u}}\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}} - \nabla_{[\mathbf{u}, \mathbf{v}]}. \quad (19)$$

Definition 4 Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \sec TM$ and $\alpha \in \sec \bigwedge^1 T^*M$. The torsion and curvature tensors of a connection ∇ are the mappings $\mathbf{T} : \sec(\bigwedge^1 T^*M \otimes TM \otimes TM) \rightarrow \mathbb{R}$ and $\mathbf{R} : \sec(TM \otimes \bigwedge^1 T^*M \otimes TM \otimes TM) \rightarrow \mathbb{R}$ given by

²See, e.g., [13].

$$\mathbf{T}(\alpha, \mathbf{u}, \mathbf{v}) = \alpha(\tau(\mathbf{u}, \mathbf{v})), \quad (20)$$

$$\mathbf{R}(\mathbf{w}, \alpha, \mathbf{u}, \mathbf{v}) = \alpha(\rho(\mathbf{u}, \mathbf{v})\mathbf{w}), \quad (21)$$

In an arbitrary basis $\langle e_\mu \rangle$ for $T(U \cap V \cap W)$ and dual basis $\langle \theta^\mu \rangle$ for $T^*(U \cap V \cap W)$ we have

$$\mathbf{T}(\theta^\lambda, e_\mu, e_\nu) := \mathbf{T}_{\mu\nu}^{\lambda\cdot\cdot} = \gamma_{\cdot\mu\nu}^{\lambda\cdot\cdot} - \gamma_{\cdot\nu\mu}^{\lambda\cdot\cdot} - \mathbf{c}_{\cdot\nu\mu}^{\lambda\cdot\cdot}. \quad (22)$$

$$\begin{aligned} \mathring{\mathbf{R}}_{\mu\cdot\alpha\beta}^{\lambda\cdot\cdot} &:= \mathring{\mathbf{R}}(e_\mu, \theta^\lambda, e_\alpha, e_\beta) \\ &= e_\alpha(\mathring{\gamma}_{\cdot\beta\mu}^{\lambda\cdot\cdot}) - e_\beta(\mathring{\gamma}_{\cdot\alpha\mu}^{\lambda\cdot\cdot}) - \mathring{\gamma}_{\cdot\alpha\kappa}^{\lambda\cdot\cdot} \mathring{\gamma}_{\cdot\beta\mu}^{\kappa\cdot\cdot} - \mathring{\gamma}_{\cdot\kappa\beta}^{\lambda\cdot\cdot} \mathring{\gamma}_{\cdot\alpha\mu}^{\kappa\cdot\cdot} - \mathbf{c}_{\cdot\alpha\beta}^{\kappa\cdot\cdot} \mathring{\gamma}_{\cdot\kappa\mu}^{\lambda\cdot\cdot}, \\ \mathbf{R}_{\mu\cdot\alpha\beta}^{\lambda\cdot\cdot} &:= \mathbf{R}(e_\mu, \theta^\lambda, e_\alpha, e_\beta) \\ &= e_\alpha(\gamma_{\cdot\beta\mu}^{\lambda\cdot\cdot}) - e_\beta(\gamma_{\cdot\alpha\mu}^{\lambda\cdot\cdot}) - \gamma_{\cdot\alpha\kappa}^{\lambda\cdot\cdot} \gamma_{\cdot\beta\mu}^{\kappa\cdot\cdot} - \gamma_{\cdot\kappa\beta}^{\lambda\cdot\cdot} \gamma_{\cdot\alpha\mu}^{\kappa\cdot\cdot} - \mathbf{c}_{\cdot\alpha\beta}^{\kappa\cdot\cdot} \gamma_{\cdot\kappa\mu}^{\lambda\cdot\cdot}. \end{aligned} \quad (23)$$

2.2.1 Relation Between the Curvature Tensors of D and \mathring{D}

The components of the curvature tensors relative to the coordinate basis associated to the coordinates $\langle x^\mu \rangle$ covering V are:

$$R_{\mu\cdot\alpha\beta}^{\lambda\cdot\cdot} = \mathring{R}_{\mu\cdot\alpha\beta}^{\lambda\cdot\cdot} + J_{\mu\cdot[\alpha\beta]}^{\lambda\cdot\cdot}, \quad (24)$$

where

$$\begin{aligned} J_{\mu\cdot\alpha\beta}^{\lambda\cdot\cdot} &:= \mathring{D}_\alpha K_{\beta\mu}^{\lambda\cdot\cdot} - K_{\cdot\alpha\mu}^{\sigma\cdot\cdot} K_{\cdot\beta\sigma}^{\lambda\cdot\cdot}, \\ J_{\mu\cdot[\alpha\beta]}^{\lambda\cdot\cdot} &= J_{\mu\cdot\alpha\beta}^{\lambda\cdot\cdot} - J_{\mu\cdot\beta\alpha}^{\lambda\cdot\cdot}. \end{aligned} \quad (25)$$

We need also the

Proposition 5 *Let $\mathbf{Z} \in \sec TV$ be a timelike vector field such that $\mathbf{g}(\mathbf{Z}, \mathbf{Z}) = 1$. Then, there exist, in a coordinate neighborhood V , three spacelike vector fields \mathbf{e}_i which together with \mathbf{Z} form an orthogonal moving frame for $x \in V \subset M$ [3].*

Proof. Suppose that the metric of the manifold in a chart (ψ, V) with coordinate functions $\langle x^\mu \rangle$ is $\mathbf{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu$. Let $\mathbf{Z} = (Q^\mu \partial / \partial x^\mu) \in \sec TV$ be an arbitrary reference frame and $\alpha_{\mathbf{Z}} = \mathbf{g}(\mathbf{Z}, \cdot) = Z_\mu dx^\mu$, $Z_\mu = g_{\mu\nu} Z^\nu$. Then, $g_{\mu\nu} Z^\mu Z^\nu = 1$. Now, define

$$\begin{aligned} \theta^0 &= (\alpha_{\mathbf{Z}})_\mu dx^\mu = Z_\mu dx^\mu, \\ \gamma_{\mu\nu} &= g_{\mu\nu} - Z_\mu Z_\nu. \end{aligned} \quad (26)$$

Then the metric \mathbf{g} can be written due to the hyperbolicity of the manifold as

$$\begin{aligned} \mathbf{g} &= \eta_{\mu\nu} \theta^\mu \otimes \theta^\nu, \\ \sum_{i=1}^3 \theta^i \otimes \theta^i &= \gamma_{\mu\nu}(x) dx^\mu \otimes dx^\nu. \end{aligned} \quad (27)$$

Now, call $e_0 = Z$ and take e_i such that $\theta^i(e_j) = \delta_j^i$. It follows immediately that $g(e_\mu, e_\nu) = \eta_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$. ■

3 Observers and Reference Frames

Definition 6 An observer in a Lorentzian structure $\langle M, g \rangle$ is a timelike curve γ pointing to the future such that $g(\gamma_*, \gamma_*) = 1$.

Definition 7 A reference frame in $U \cap V \cap W \subset M$ in a Lorentzian structure $\langle M, g \rangle$ is a timelike vector field Z ($g(Z, Z) = 1$) such that each one of its integral lines is an observer.

So, if σ is an integral line of Z , its parametric equations are

$$\frac{d(x^\mu \circ \sigma(\tau))}{d\tau} = Z^\mu(x^\alpha(\tau)). \quad (28)$$

Definition 8 A naturally adapted coordinate system $\langle x^\mu \rangle$ to a reference frame $Z \in \text{sec} TV$ (denoted $\langle \text{nacs}|Z \rangle$) is one where the spacelike components of Z are null. Note that such a chart always exist [2].

Remark 9 The definition of a reference frame in a Lorentzian spacetime or in a Riemann-Cartan spacetime is the same as above since that definition does not depends on the additional objects entering these structures.

3.1 References Frames in $\langle M, g, \mathring{D}, \tau_g, \uparrow \rangle$ and $\langle M, g, D, \tau_g, \uparrow \rangle$

Given a reference frame Z in $U \cap V \cap W \subset M$, consider the physically equivalent 1-form field

$$\alpha = g(Z, \cdot). \quad (29)$$

Then we have:

$$\mathring{D}\alpha = \mathring{a} \otimes \alpha + \mathring{\omega} + \mathring{\sigma} + \frac{1}{3}\mathring{\mathfrak{E}}h, \quad (30)$$

where

$$h = g - \alpha \otimes \alpha \quad (31)$$

is the projection tensor, α is the (form) *acceleration* of Z , $\mathring{\omega}$ is the *rotation* tensor (or *vortex*) of Z , $\mathring{\sigma}$ is the *shear* of Z and $\mathring{\mathfrak{E}}$ is the *expansion ratio* of Z . In a coordinate chart (ψ, V) with coordinate functions x^μ , writing $Z = Z^\mu \partial / \partial x^\mu$ and $h = (g_{\mu\nu} - Z_\mu Z_\nu) dx^\mu \otimes dx^\nu$ we have

$$\begin{aligned} \mathring{a} &= g(\mathring{D}_Z Z, \cdot) = \mathring{D}_Z \alpha \\ \mathring{\omega}_{\alpha\beta} &= Z_{[\mu;\nu]} h_\alpha^\mu h_\beta^\nu, \\ \mathring{\sigma}_{\alpha\beta} &= [Z_{(\mu;\nu)} - \frac{1}{3}\mathring{\mathfrak{E}}h_{\mu\nu}] h_\alpha^\mu h_\beta^\nu, \\ \mathring{\mathfrak{E}} &= \mathring{D}_\mu Z^\mu. \end{aligned} \quad (32)$$

Proof. The decomposition given by Eq.(30) can be trivially verified if we use an orthonormal basis where $e_0 = \mathbf{Z}$, for in this case $\alpha = \theta^0$ and we realize that

$$\begin{aligned}\dot{\omega}_{ij} &= -\frac{1}{2} (\dot{\gamma}_{\cdot ij}^{0\cdot\cdot} - \dot{\gamma}_{\cdot ji}^{0\cdot\cdot}) = -\frac{1}{2} c_{ij}^{0\cdot\cdot}, \\ \dot{\sigma}_{ij} &= -\frac{1}{2} (\dot{\gamma}_{\cdot ij}^{0\cdot\cdot} + \dot{\gamma}_{\cdot ji}^{0\cdot\cdot}) - \frac{1}{3} \mathfrak{E} h_{ij}, \\ \mathfrak{E} &= -\eta^{ij} \dot{\gamma}_{\cdot ij}^{0\cdot\cdot}.\end{aligned}\tag{33}$$

■

Remark 10 We can show that the vorticity tensor has the same components as the object

$$g(\star(\alpha \wedge d\alpha), \cdot),\tag{34}$$

where \star is the Hodge star operator. Indeed, we have

$$\begin{aligned}\star(\alpha \wedge d\alpha) &= \star(\theta^0 \wedge d\theta^0) = -\frac{1}{2} c_{ij}^{0\cdot\cdot} \star(\theta^0 \wedge \theta^i \wedge \theta^j) \\ &= -c_{23}^{0\cdot\cdot} \theta_1 + c_{13}^{0\cdot\cdot} \theta_2 - c_{12}^{0\cdot\cdot} \theta_3,\end{aligned}$$

and

$$g(\star(\alpha \wedge d\alpha), \cdot) = c_{23}^{0\cdot\cdot} e_1 + c_{31}^{0\cdot\cdot} e_2 + c_{12}^{0\cdot\cdot} e_3 = \frac{1}{2} \epsilon^{0ijk} c_{jk}^{0\cdot\cdot} e_i.\tag{35}$$

Remark 11 Eq.(32) is the basis for the classification of reference frames in a Lorentzian spacetime structure³ [15, 18, 22] and in order to be possible to talk about the classification of reference frames in a Riemann-Cartan spacetime structure we need the

Proposition 12

$$D\alpha = a \otimes \alpha + \omega + \sigma + \frac{1}{3} \mathfrak{E} h,\tag{36}$$

$$a = D_Z \alpha,\tag{37a}$$

$$\omega = \dot{\omega} + T^0, \quad \sigma = \dot{\sigma} + \frac{1}{3} (\mathfrak{E} - \mathfrak{E}) h + S^0,\tag{37b}$$

$$T^0 = \frac{1}{2} T_{\cdot ij}^{0\cdot\cdot} \theta^i \wedge \theta^j, \quad S^0 = -\frac{1}{2} S_{\cdot ij}^{0\cdot\cdot} \theta^i \otimes \theta^j.\tag{37c}$$

Proof. It is a simple exercise using an orthonormal basis where $\alpha = \theta^0$. ■

Remark 13 We observe that in a Riemann-Cartan spacetime the interpretation of ω (in the decomposition of $D\alpha$ given by Eq.(36)) is the same as $\dot{\omega}$ in a Lorentzian spacetime [18], i.e., it measures the rotation that one of the infinitesimally nearby curves to an integral curve γ (an ‘observer’) of \mathbf{Z} had

³For the classification of reference frames in a Newtonian spacetime structure, see [16].

in an infinitesimal lapse of proptime with relation to an orthonormal basis Fermi-transported by the ‘observer’ γ . The interpretation of the terms σ and \mathfrak{E} are also analogous to the corresponding terms in a Lorentzian spacetime. Thus, a reference frame is non-rotating if $\omega = 0$, i.e., $\dot{\omega} = -\mathbf{T}^0$ and Eq.(37c) shows that torsion is indeed related to rotation from the point of view of a Lorentzian spacetime structure.

3.2 Inertial Reference Frames in $\langle M \simeq \mathbb{R}^4, \eta, \overset{m}{D}, \tau_\eta, \uparrow \rangle$

Now, let $\langle M, \mathbf{g}, \overset{\circ}{D}, \tau_{\mathbf{g}}, \uparrow \rangle = \langle M \simeq \mathbb{R}^4, \eta, \overset{m}{D}, \tau_\eta, \uparrow \rangle$ and let $\langle \mathbf{x}^\mu \rangle$ be coordinates in the Einstein-Lorentz-Poincaré gauge for M . If the matrix with entries $\eta_{\mu\nu}$ is the diagonal matrix $\text{diag}(1, -1, -1, -1)$, we have

$$\eta = \eta_{\mu\nu} d\mathbf{x}^\mu \otimes d\mathbf{x}^\nu \quad (38)$$

If we put $\mathbf{I} = \partial/\partial \mathbf{x}^0$ we see immediately that that $\langle \mathbf{x}^\mu \rangle$ is a $\langle nacs|\mathbf{Z} \rangle$. We have trivially

$$\overset{m}{D}\alpha_{\mathbf{I}} = 0, \quad (39)$$

which means that for the reference frame $\mathbf{I} = \partial/\partial \mathbf{x}^0$ we have $\mathbf{a} = 0$, $\omega = 0$, $\sigma = 0$, $\mathfrak{E} = 0$.

Definition 14 *A inertial reference frame (IRF) in $\langle M \simeq \mathbb{R}^4, \eta, \overset{m}{D}, \tau_\eta, \uparrow \rangle$ is reference frame \mathbf{I} such that $\overset{m}{D}\alpha_{\mathbf{I}} = 0$.*

So, inertial reference frames in Special Relativity are not accelerating, not rotating, have no shear and no deformation. Of course, since $\overset{m}{D}_{\partial/\partial \mathbf{x}^0} \partial/\partial \mathbf{x}^0 = 0$, each one of the integral lines of the vector field $\mathbf{I} = \partial/\partial \mathbf{x}^0$ is a timelike autoparallel (in this case, also a geodesic) of Minkowski spacetime (a straight line).

3.3 Is there IRFs in Lorentzian and Riemann-Cartan Spacetimes?

The answer is yes for the Lorentzian case only if we can find a reference frame \mathbf{I} such that $\overset{\circ}{D}\alpha_{\mathbf{I}} = 0$. In general this equation has no solution in a general $\langle M, \mathbf{g}, \overset{\circ}{D}, \tau_{\mathbf{g}}, \uparrow \rangle$ structure and indeed we have the

Proposition 15 [22] *An IRF exists in the Lorentzian structure $\langle M, \mathbf{g}, \overset{\circ}{D}, \tau_{\mathbf{g}}, \uparrow \rangle$ only if the Ricci tensor satisfies*

$$\text{Ricci}(\mathbf{I}, \mathbf{Y}) = 0 \quad (40)$$

for any $\mathbf{Y} \in \text{sec } TM$.

Remark 16 This excludes, e.g., Friedmann universe spacetimes, Einstein-de Sitter spacetime. So, no IRF exist in many models of GRT considered to be of interest by one reason or another by ‘professional relativists’.

Remark 17 The situation in a Riemann-Cartan spacetime is more complicated and will be analyzed elsewhere, but we observe that in an arbitrary teleparallel spacetime structure $\langle M, \mathbf{g}, \overset{e}{\nabla}, \tau_{\mathbf{g}}, \uparrow \rangle$ the teleparallel basis $\langle \mathbf{e}_\mu \rangle$ satisfies $\nabla_{\mathbf{e}_\nu} \mathbf{e}_\mu = 0$. Then the reference frame \mathbf{e}_0 is a **IRF**.

3.4 Pseudo Inertial Reference Frames

Definition 18 A reference frame $\mathfrak{J} \in \text{sec } TU$, $U \subset M$ is said to be a pseudo inertial reference frame (**PIRF**) if $D_{\mathfrak{J}}\mathfrak{J} = 0$, $\alpha_{\mathfrak{J}} \wedge d\alpha_{\mathfrak{J}} = 0$ and $\alpha_{\mathfrak{J}} = g(\mathfrak{J}, \cdot)$.

This definition means that a **PIRF** is in free fall and it is non rotating. It means also that it is at least *locally synchronizable*, but we are not going to discuss synchronizability here (details may be found, e.g., in ([18])).

4 What is a LIRF in $\langle M, \mathbf{g}, \nabla, \tau_{\mathbf{g}}, \uparrow \rangle$

4.1 Normal Coordinate Functions at $p_o \in M$

In what follows ∇ denotes $\overset{\circ}{D}$ or D . We will specialize our discourse at appropriate places.

Let (φ, U) be a local chart around $p_o \in U$ with coordinate functions $\langle \xi^\mu \rangle$. Let $\gamma : \mathbb{R} \supset I \rightarrow M$, $\tau \mapsto \gamma(\tau)$ an *autoparallel*⁴ in M according to an arbitrary connection ∇ , i.e.,

$$\nabla_{\gamma_*} \gamma_* = 0. \quad (41)$$

Take arbitrary points $p_o, q \in M$. Put

$$\gamma(0) = p_o, \quad \xi_q := \left. \frac{d}{d\tau} \right|_{p_o} = \xi_q^\mu \left. \frac{\partial}{\partial \xi^\mu} \right|_{p_o} = \xi_q^\mu \mathbf{e}_\mu \in T_{p_o}M, \quad (42)$$

$$\xi^\mu(p_o) = \xi_{p_o}^\mu = 0, \quad \xi^\mu(q) = \xi_q^\mu \neq 0 \quad (43)$$

Although the notation looks strange it will become clear in a while. Now, any autoparallel emanating from p_o is specified by a given $\xi_q \in T_{p_o}M$. Indeed, take q ‘near’ p_o , this statement simply meaning here that the coordinates difference $\Delta \xi^\mu = \xi_q^\mu - \xi_{p_o}^\mu = \xi_q^\mu \ll 1$. In general there may be many autoparallels that connect p_o to q . However, there exists a unique autoparallel $\gamma_q : \mathbb{R} \supset I \rightarrow M$, $\tau \mapsto \gamma_q(\tau)$ such that

$$\gamma_q(0) = p_o, \quad \gamma_q(1) = q. \quad (44)$$

So, under the above conditions we see that if $\xi_q^\mu \ll 1$, then q uniquely specifies a vector $\xi_q = \xi_q^\mu \mathbf{e}_\mu \in T_{p_o}M$. It is evident that $\varphi : q \mapsto \xi_q$ serves as a good coordinate system in a neighbourhood of p_o . We have

⁴Autoparallels in a Lorentzian spacetime structure coincide with geodesics.

Definition 19 $\varphi : q \mapsto (\xi_q^0, \xi_q^1, \xi_q^2, \xi_q^3) := \{\xi_q^\mu\}$ is called a normal coordinate chart based on p_o $[(\text{nccb}|p_o)]$ with basis $\mathbf{e}_\mu = \frac{\partial}{\partial \xi^\mu} \Big|_{p_o}$.

Obviously $\varphi(p_o) = (0, 0, 0, 0)$.

Definition 20 The so-called exponential map is the mapping

$$\begin{aligned} \exp : T_{p_o}M &\rightarrow M, & \exp \boldsymbol{\xi}_q &= q, \\ \varphi(\exp \boldsymbol{\xi}_q) &= \{\xi_q^\mu\}. \end{aligned} \quad (45)$$

With respect to the $(\text{nccb}|p_o)$ an autoparallel $\gamma(\tau)$ with $\gamma(0) = p_o$ and $\gamma(1) = q$ is represented by

$$\varphi(\gamma(\tau)) = \{\xi_q^\mu \tau\}. \quad (46)$$

4.2 Autoparallels Passing Trough p_o in $\langle M, \mathbf{g}, \mathring{D}, \boldsymbol{\tau}_g, \uparrow \rangle$

If $\gamma(\tau)$ is an autoparallel in the structure $\langle M, \mathbf{g}, \mathring{D}, \boldsymbol{\tau}_g, \uparrow \rangle$ it satisfies the equation $\mathring{D}_{\gamma_*} \gamma_* = 0$. Let

$$\mathring{D}_{\partial/\partial \xi^\mu} \partial/\partial \xi^\nu := \mathring{\Gamma}_{\cdot\mu\nu}^{\alpha\cdot\cdot} \partial/\partial \xi^\alpha. \quad (47)$$

We shall prove that the connection coefficients $\mathring{\Gamma}_{\cdot\mu\nu}^{\alpha\cdot\cdot}$ vanishes at p_o . Indeed, the coordinate expression of the autoparallel equation $\mathring{D}_{\gamma_*} \gamma_* = 0$ is

$$\begin{aligned} \mathring{D}_{\gamma_*} \gamma_* &= \mathring{D}_{\frac{d}{d\tau}} \left[\frac{d\xi^\nu}{d\tau} \partial/\partial \xi^\nu \right] = \frac{d\xi^\mu}{d\tau} \mathring{D}_{\partial/\partial \xi^\mu} \frac{d\xi^\nu}{d\tau} (\tau) \partial/\partial \xi^\nu \\ &= \frac{d^2 \xi^\nu}{d\tau^2} \partial/\partial \xi^\nu + \frac{d\xi^\nu}{d\tau} \frac{d\xi^\mu}{d\tau} \mathring{D}_{\partial/\partial \xi^\mu} \partial/\partial \xi^\nu \\ &= \frac{d^2 \xi^\nu}{d\tau^2} \partial/\partial \xi^\nu + \frac{d\xi^\nu}{d\tau} \frac{d\xi^\mu}{d\tau} [\mathring{\Gamma}_{\cdot\mu\nu}^{\alpha\cdot\cdot} (\varphi(\gamma(\tau))) \partial/\partial \xi^\alpha] \\ &= \left\{ \frac{d^2 \xi^\nu}{d\tau^2} + \frac{d\xi^\alpha}{d\tau} \frac{d\xi^\mu}{d\tau} [\mathring{\Gamma}_{\cdot\mu\alpha}^{\nu\cdot\cdot} (\varphi(\gamma(\tau)))] \right\} \partial/\partial \xi^\nu = 0. \end{aligned} \quad (48)$$

Now, since at any point q' near p_o it is

$$\xi^\nu(\tau) = \xi_{q'}^\nu \tau \quad (49)$$

we have $\frac{d^2 \xi^\nu}{d\tau^2} \Big|_{\tau=0} = 0$ and Eq.(48) gives immediately in view of the fact that $\mathring{\Gamma}_{\cdot\mu\nu}^{\alpha\cdot\cdot} = \mathring{\Gamma}_{\cdot\nu\mu}^{\alpha\cdot\cdot}$ that

$$\mathring{\Gamma}_{\cdot\mu\alpha}^{\nu\cdot\cdot} (\varphi(\gamma(0))) = 0. \quad (50)$$

Remark 21 In what follows for simplicity of notation and when no confusion arises we eventually use the sloop notation $\mathring{\Gamma}_{\cdot\mu\alpha}^{\nu\cdot\cdot} (p_o) := \mathring{\Gamma}_{\cdot\mu\alpha}^{\nu\cdot\cdot} (\varphi(\gamma(0)))$.

Since it is well known that

$$\mathring{\Gamma}_{\cdot\mu\alpha}^{\nu\cdot\cdot} = \frac{1}{2}g^{\nu\kappa} \left(\frac{\partial g_{\kappa\alpha}}{\partial \xi^\mu} + \frac{\partial g_{\kappa\mu}}{\partial \xi^\alpha} - \frac{\partial g_{\mu\alpha}}{\partial \xi^\kappa} \right), \quad (51)$$

we arrive at the conclusion that at the $p_o = \gamma(0)$ we can choose the normal coordinate functions such that

$$\mathbf{g}(\partial/\partial \xi^\mu, \partial/\partial \xi^\nu)|_{p_o} = \eta_{\mu\nu}, \quad \left. \frac{\partial g_{\kappa\alpha}}{\partial \xi^\mu} \right|_{p_o} = 0. \quad (52)$$

We can also show through a simple computation that for any $q' \in U$, $q' \notin p_o$ we have

$$\mathring{\Gamma}_{\cdot\beta\gamma,\mu}^{\alpha\cdot\cdot}(\xi^\mu)|_{q'} = \partial/\partial \xi^\mu \mathring{\Gamma}_{\cdot\beta\gamma}^{\alpha\cdot\cdot}(\xi^\mu)|_{q'} = -\frac{1}{3}(\mathring{R}_{\beta\cdot\gamma\mu}^{\alpha\cdot\cdot}(\xi^\mu) + \mathring{R}_{\gamma\cdot\beta\mu}^{\alpha\cdot\cdot}(\xi^\mu))|_{q'}, \quad (53)$$

and also that for the chart (ψ, V) with coordinate functions $\langle x^\mu \rangle$ we have for q' near p_o

$$\begin{aligned} \xi^\mu &= x^\mu + \frac{1}{2}\mathring{\Gamma}_{\cdot\alpha\beta}^{\mu\cdot\cdot}(p_o)x^\alpha x^\beta, \\ x^\mu &= \xi^\mu - \frac{1}{2}\mathring{\Gamma}_{\cdot\alpha\beta}^{\mu\cdot\cdot}(p_o)\xi^\alpha \xi^\beta, \end{aligned} \quad (54)$$

where $\mathring{\Gamma}_{\cdot\alpha\beta}^{\mu\cdot\cdot}(p_o)$ denotes the values of the connection coefficients in the coordinates $\langle x^\mu \rangle$, i.e.,

$$\mathring{D}_{\partial/\partial x^\mu} \partial/\partial x^\nu = \mathring{\Gamma}_{\cdot\mu\nu}^{\alpha\cdot\cdot} \partial/\partial x^\alpha. \quad (55)$$

Remark 22 Let $\gamma \in U \subset M$ be the world line of an observer in autoparallel motion in spacetime, i.e., $\mathring{D}_{\gamma_*} \gamma_* = 0$. Then the above developments show that we can introduce in U normal coordinate functions $\langle \xi^\mu \rangle$ such that for every $p \in \gamma$ we have

$$\begin{aligned} \left. \frac{\partial}{\partial \xi^0} \right|_{p \in \gamma} &= \gamma_*|_p, \quad \mathbf{g}(\partial/\partial \xi^\mu, \partial/\partial \xi^\nu)|_{p \in \gamma} = \eta_{\mu\nu}, \quad \left. \frac{\partial g_{\kappa\alpha}}{\partial \xi^\mu} \right|_p = 0 \\ \mathring{\Gamma}_{\cdot\nu\rho}^{\mu\cdot\cdot}(\xi^\mu)|_{p \in \gamma} &= g^{\mu\alpha} \mathbf{g}(\partial/\partial \xi^\alpha, \mathring{D}_{\partial/\partial \xi^\nu} \partial/\partial \xi^\rho)|_{p \in \gamma} = 0. \end{aligned} \quad (56)$$

Finally observe that

$$\mathring{R}_{\mu\cdot\alpha\beta}^{\cdot\lambda\cdot\cdot} \Big|_{p \in \gamma} = \frac{\partial \mathring{\Gamma}_{\cdot\beta\mu}^{\lambda\cdot\cdot}}{\partial \xi^\alpha} \Big|_{p \in \gamma} - \frac{\partial \mathring{\Gamma}_{\cdot\alpha\mu}^{\lambda\cdot\cdot}}{\partial \xi^\beta} \Big|_{p \in \gamma} \quad (57)$$

which is non null if the curvature tensor is non null in U .

4.3 LIRFs in $\langle M, g, \mathring{D}, \tau_g, \uparrow \rangle$

Definition 23 Given a timelike autoparallel line $\gamma \subset U \subset M$ and coordinates $\langle \xi^\mu \rangle$ covering U we say that a reference frame $\mathbf{L} = \partial/\partial \xi^0 \in \text{sec } TU$ is a local inertial Lorentz reference frame associated to γ (**LIRF** $_\gamma$)⁵ iff

$$\begin{aligned} \mathbf{L}|_{p \in \gamma} &= \frac{\partial}{\partial \xi^0} \Big|_{p \in \gamma} = \gamma_*|_p, \\ \alpha_{\mathbf{L}} \wedge d\alpha_{\mathbf{L}}|_{p \in \gamma} &= 0, \\ g(\partial/\partial \xi^\mu, \partial/\partial \xi^\nu)|_{p \in \gamma} &= \eta_{\mu\nu}, \quad \frac{\partial g_{\alpha\beta}}{\partial \xi^\mu} \Big|_{p \in \gamma} = 0. \end{aligned} \quad (58)$$

Moreover, we say also that the normal coordinate functions (also called in Physics textbooks local Lorentz coordinate functions) $\langle \xi^\mu \rangle$ are *associated* with the **LIRF** $_\gamma$.

Remark 24 It is very important to have in mind that for a **LIRF** $_\gamma$ \mathbf{L} , in general $\mathring{D}_{\mathbf{L}}\mathbf{L}|_{p \notin \gamma} \neq 0$ (i.e., only the integral line γ of \mathbf{L} is in free fall in general), and also eventually $\alpha_{\mathbf{L}} \wedge d\alpha_{\mathbf{L}}|_{p \notin \gamma} \neq 0$, which may be a surprising result. In contrast, a **PIRF** \mathfrak{J} such that $\mathfrak{J}|_\gamma = \mathbf{L}|_\gamma$ has all its integral lines in free fall and the rotation of the frame is always null in all points where the frame is defined. Finally it is worth to recall that both \mathfrak{J} and \mathbf{L} may eventually have shear and expansion even at the points of the autoparallel line γ that they have in common. More details in [18].

Let γ be an autoparallel line as in definition 23. A section s of the orthogonal frame bundle $FU, U \subset M$ is called an *inertial moving frame along γ* (**IMF** $_\gamma$) when the set

$$s_\gamma = \{(e_0(p), e_1(p), e_2(p), e_3(p)), p \in \gamma \cap U\} \subset s, \quad (59)$$

it such that $\forall p \in \gamma$

$$e_0(p) = \gamma_*|_{p \in \gamma}, \quad \mathbf{g}(e_\mu, e_\nu)|_{p \in \gamma} = \eta_{\mu\nu}, \quad \frac{\partial g_{\alpha\beta}}{\partial \xi^\mu} \Big|_{p \in \gamma} = 0. \quad (60)$$

wich implies

$$\mathring{\Gamma}_{\cdot\nu\rho}^{\mu\cdot\cdot}(p) = g^{\mu\alpha} \mathbf{g}(e_\alpha(p), \mathring{D}_{e_\nu(p)} e_\rho(p)) = 0, \forall p \in \gamma. \quad (61)$$

Remark 25 The existence of $s \in \text{sec } FU$ satisfying the above conditions can be easily proved. Introduce coordinate functions $\langle \xi^\mu \rangle$ for U such that at $p_o \in \gamma$, $e_0(p_o) = \frac{\partial}{\partial \xi^0} \Big|_{p_o} = \gamma_*|_{p_o}$, and $e_i(p_o) = \frac{\partial}{\partial \xi^i} \Big|_{p_o}$, $i = 1, 2, 3$ (three orthonormal vectors) satisfying Eq.(60) and parallel transport the set $e_\mu(p_o)$ along γ . The set $e_\mu(p_o)$ will then also be Fermi transported since γ is a geodesic and as such they define the standard of no rotation along γ . See details in [18].

⁵When no confusion arises and γ is clear from the context we simply write **LIRF**.

Remark 26 Let $\mathfrak{J} \in \sec TV$ be a **PIRF** and $\gamma \subset U \cap V$ one of its integral lines and let $\langle \xi^\mu \rangle$, $U \subset M$ be a normal coordinate system through all the points of the world line γ such that $\gamma_* = \mathfrak{J}|_\gamma$. Then, in general $\langle \xi^\mu \rangle$ is not a (nacs| \mathfrak{J}) in U , i.e., $\mathfrak{J}|_{p \notin \gamma} \neq \partial/\partial \xi^0|_{p \notin \gamma}$ even if $\mathfrak{J}|_{p \in \gamma} = \partial/\partial \xi^0|_{p \in \gamma}$.

Remark 27 It is very much important to recall that a reference frame field as introduced above is a mathematical instrument. It did not necessarily need to have a material substratum (i.e., to be realized as a material physical system) in the points of the spacetime manifold where it is defined. More properly, we state that the integral lines of the vector field representing a given reference frame do not need to correspond to worldlines of real particles. If this crucial aspect is not taken into account we may incur in serious misunderstandings.

Remark 28 Physics textbooks and even most of the professional articles in GR do not distinguish between the very different concepts of reference frames, coordinate systems, sections of the frame bundle and does not leave clear what is meant by the word local. In general what authors mean by a local inertial reference system is the concept of normal coordinates associated to a timelike autoparallel curve γ as describe above. Moreover, keep in mind that of course, $\gamma_* = \frac{d}{d\tau} = \frac{\partial}{\partial \xi^0}|_\gamma$.

4.4 LIRFs in $\langle M, g, D, \tau_g, \uparrow \rangle$

We have seen above that we can always introduce around a point $p_o \in U \subset M$ in a Lorentzian $\langle M, g, \overset{\circ}{D}, \tau_g, \uparrow \rangle$ or in a Riemann-Cartan $\langle M, g, D, \tau_g, \uparrow \rangle$ structure a chart (φ, U) with normal coordinate functions.

However, it is not licit a priori to assume that the normal coordinate functions of the two structures coincide. So, we denote by $\langle \zeta^\mu \rangle$ the Riemann-Cartan normal coordinate functions around p_o in what follows. In the case of a Lorentzian structure we found that at p_o the connection coefficients

$$\overset{\circ}{\Gamma}^{\alpha\cdots}_{\mu\nu}(p_o) = (\overset{\circ}{D}_{\partial/\partial \xi^\mu} \partial/\partial \xi^\nu) \cdot (g^{\alpha\kappa} \partial/\partial \xi^\kappa) = 0.$$

However, we are not going to suppose that this is generally the case in a Riemann-Cartan structure. So, let us investigate which conditions

$$\Gamma^{\alpha\cdots}_{\mu\nu}(p_o) = (D_{\partial/\partial \zeta^\mu} \partial/\partial \zeta^\nu) \cdot (g^{\alpha\kappa} \partial/\partial \zeta^\kappa), \quad (62)$$

must satisfy in normal coordinates $\langle \zeta^\mu \rangle$. A Riemann-Cartan autoparallel γ passing through p_o and neighboring points q' (in the sense mentioned above), satisfy $D_{\gamma_*} \gamma_* = 0$, and we have

$$\frac{d^2 \zeta^\nu}{d\tau^2} + \frac{d\zeta^\alpha}{d\tau} \frac{d\zeta^\mu}{d\tau} [\Gamma^{\nu\cdots}_{\mu\alpha}(\varphi(\gamma(\tau)))] = 0. \quad (63)$$

If the autoparallel equation is for points from p_o to q given by $\zeta^\nu(\tau) = \zeta^\nu_{q'} \tau$ (recall Eq.(49)) then since $\frac{d^2 \zeta^\nu}{d\tau^2} \Big|_{\tau=0} = 0$, at p_o we must have $\Gamma^{\nu\cdots}_{(\mu\alpha)}(p_o) = \frac{1}{2} (\Gamma^{\nu\cdots}_{\mu\alpha}(p_o) + \Gamma^{\nu\cdots}_{\alpha\mu}(p_o)) = 0$, i.e.,

$$\mathbf{\Gamma}_{\cdot\mu\alpha}^{\nu\cdots}(p_o) = -\mathbf{\Gamma}_{\cdot\alpha\mu}^{\nu\cdots}(p_o). \quad (64)$$

Now, if we recall Eq.(14), Eq.(16), Eq.(17) which gives the components of the torsion and strain tensors, we see that in the case of normal coordinates $\langle\zeta^\mu\rangle$ we must have

$$\mathbf{T}_{\cdot\mu\alpha}^{\nu\cdots}(p_o) = 2\mathbf{\Gamma}_{\cdot\mu\alpha}^{\nu\cdots}(p_o), \quad (65a)$$

$$\mathbf{S}_{\cdot\mu\alpha}^{\nu\cdots}(p_o) = -2\mathbf{\dot{\Gamma}}_{\cdot\mu\alpha}^{\nu\cdots}(p_o), \quad (65b)$$

which are the conditions that select the normal coordinate functions $\langle\zeta^\mu\rangle$ near p_o in a Riemann-Cartan spacetime.

Remark 29 We did not suppose, of course, that the autoparallels of the Levi-Civita and Riemann-Cartan connections coincide (since this is trivially false)⁶. So, we have the question: when does the two kinds of autoparallels coincide? If they do coincide then the Lorentzian and Riemann-Cartan normal coordinate functions around p_o must coincide and since for a autoparallel from p_o to q , it is $\left.\frac{d^2\xi^\nu}{d\tau^2}\right|_{\tau=0} = 0$ we must have again that $\mathbf{\Gamma}_{\cdot\mu\alpha}^{\nu\cdots}(p_o) = -\mathbf{\Gamma}_{\cdot\alpha\mu}^{\nu\cdots}(p_o)$. But now since $\mathbf{\dot{\Gamma}}_{\cdot\alpha\beta}^{\nu\cdots}(p_o) = 0$ we arrive at the conclusion that

$$\mathbf{T}_{\cdot\mu\alpha}^{\nu\cdots} = 2\mathbf{\Gamma}_{\cdot\mu\alpha}^{\nu\cdots}(p_o), \quad (66a)$$

$$\mathbf{S}_{\cdot\mu\alpha}^{\nu\cdots}(p_o) = 0. \quad (66b)$$

Eq.(66b) implies moreover that $\mathbf{T}_{\nu\mu\alpha}^{\cdots}(p_o) = -\mathbf{T}_{\mu\nu\alpha}^{\cdots}(p_o) = -\mathbf{T}_{\alpha\mu\nu}^{\cdots}(p_o)$, i.e., the torsion tensor must be completely anti-symmetric at all manifold points (since p_o is arbitrary):

$$\mathbf{T}_{\mu\alpha\nu}^{\cdots}(p_o) = \mathbf{T}_{[\mu\alpha\nu]}^{\cdots}(p_o) \quad (67)$$

Eq.(67) is then the condition for the two kinds of autoparallel to coincide. It is a very particular condition and contrary to what is stated in [4, 5, 12] it is not satisfied by a general Riemann-Cartan connection and thus cannot serve the purpose of fixing coordinate functions that could model **LIRF** analogous to the ones that exist in the Lorentzian case⁷. We recall moreover that the connection coefficients of the Riemann-Cartan connection although anti-symmetric using the normal coordinate functions will be not symmetric if arbitrary coordinate functions $\langle x^\mu\rangle$ are use, since we have

$$\Gamma_{\cdot\iota\kappa}^{\lambda\cdots} = \frac{\partial x^\lambda}{\partial \xi^\mu} \frac{\partial \xi^\rho}{\partial x^\iota} \frac{\partial \xi^\sigma}{\partial x^\kappa} \mathbf{\Gamma}_{\cdot\rho\sigma}^{\mu\cdots} + \frac{\partial x^\lambda}{\partial \xi^\mu} \frac{\partial^2 \xi^\mu}{\partial x^\iota \partial x^\kappa}$$

⁶E.g., the geodesics of the Levi-Civita and the teleparallel connection on the punctured sphere \dot{S} are very different, the latter one are the so-called loxodromic spirals and the former are the maximum circles [18].

⁷Also, [21] who cites [4, 5] did not realize that total antisymmetry of the components of the torsion tensor is no more than the condition for two kinds of autoparallels (the Lorentzians and the Riemann-Cartan ones) to coincide.

The symmetric part is, of course, the same one that appears also in the transformation law for the Levi-Civita connection coefficients. We arrive at the conclusion that only for very particular spacetimes, the ones in which the strain tensor is null, we can build around a point p_o normal coordinate functions for which Eqs.(66a) and (66b) hold and it is clear that in this case $\mathbf{g}(\partial/\partial\xi^\mu, \partial/\partial\xi^\nu)|_{p_o} = \eta_{\mu\nu}$ and $\frac{\partial g_{\kappa\alpha}}{\partial\xi^\mu}\Big|_{p_o} = 0$. However, for the case of Eqs.(65a) and (65b) we cannot have $\mathbf{g}(\partial/\partial\zeta^\mu, \partial/\partial\zeta^\nu)|_{p_o} = \eta_{\mu\nu}$ and $\frac{\partial g_{\kappa\alpha}}{\partial\xi^\mu}\Big|_{p_o} = 0$, for otherwise $\overset{\circ}{\Gamma}^{\nu\cdots}_{\mu\alpha}(p_o)$ would be null.

So, in definitive normal coordinate functions are not useful to model a **LIRF** in Riemann-Cartan spacetimes. So, what can we do to model such a **LIRF** in this case?

To answer that question we need the following result:

Proposition 30 *Along any timelike autoparallel line $\gamma \subset M$ in a Riemann-Cartan spacetime structure there exists a section s of the orthogonal subframe bundle $FU \subset FM, U \subset M$ called an inertial moving frame along γ (**IMF** γ)*

$$s_\gamma = \{(\mathbf{e}_0(p), \mathbf{e}_1(p), \mathbf{e}_2(p), \mathbf{e}_3(p)), p \in U \subset M\} \subset s, \quad (68)$$

such that $\forall p \in \gamma$

$$\mathbf{e}_0|_{p \in \gamma} = \frac{1}{\sqrt{g_{00}}} \gamma_*|_{p \in \gamma}, \quad \mathbf{g}(\mathbf{e}_\mu, \mathbf{e}_\nu)|_{p \in \gamma} = \eta_{\mu\nu}, \quad (69)$$

$$[\mathbf{e}_\mu, \mathbf{e}_\nu]|_{p \in \gamma} = \mathbf{c}^{\beta\cdots}_{\mu\nu} \mathbf{e}_\beta|_{p \in \gamma}, \quad (70)$$

$$\mathbf{\Gamma}^{\mu\cdots}_{\nu\rho}|_{p \in \gamma} = 0 \quad (71)$$

and where the $\mathbf{c}^{\alpha\cdots}_{\nu\mu}$ are not all null (i.e., the $\langle \mathbf{e}_\mu \rangle$ is not a coordinate basis).

Proof. Indeed, at that any point $p \in U \subset M$ given (φ, U) , a (ncbb| p), with bases $\langle \partial/\partial\zeta^\mu \rangle$ and $\langle d\zeta^\mu \rangle$ for TU and T^*U we can find given an arbitrary vector field \mathbf{X} , a non coordinate basis $\langle \mathbf{e}_\mu \rangle$ and $\langle \boldsymbol{\theta}^\mu \rangle$ for TU and T^*U by finding a solution $\mathbf{\Lambda}$ to the matrix equation⁸,

$$0 = \mathbf{\Gamma}'_{\mathbf{X}} = \mathbf{\Lambda}_{\mathbf{X}}^{-1} \mathbf{\Gamma} \mathbf{\Lambda} + \mathbf{\Lambda}^{-1} \mathbf{X}(\mathbf{\Lambda}), \quad (72)$$

satisfying the conditions

$$\begin{aligned} [\mathbf{e}_\mu, \mathbf{e}_\nu] &= \mathbf{c}^{\beta\cdots}_{\mu\nu} \mathbf{e}_\beta, \\ \mathbf{c}^{\beta\cdots}_{\mu\nu} &= \Lambda_\mu^\alpha \mathbf{e}_\alpha(\Lambda_\nu^\beta) - \Lambda_\nu^\alpha \mathbf{e}_\alpha(\Lambda_\mu^\beta). \end{aligned} \quad (73)$$

where the $\mathbf{c}^{\beta\cdots}_{\mu\nu}$ are not all null and where the matrix function $\mathbf{\Lambda}$ with entries Λ_μ^ν is defined by

$$\begin{aligned} D_{\mathbf{X}} \mathbf{e}_\nu &:= (\mathbf{\Gamma}_{\mathbf{X}})^\mu_\nu \mathbf{e}_\mu, \quad D_{\mathbf{X}} \mathbf{e}'_\nu := (\mathbf{\Gamma}'_{\mathbf{X}})^\mu_\nu \mathbf{e}'_\mu, \\ \mathbf{e}_\mu &= \Lambda_\mu^\nu \mathbf{e}_\nu, \quad \boldsymbol{\theta}^\mu = (\Lambda^{-1})^\mu_\nu \boldsymbol{\theta}^\nu. \end{aligned} \quad (74)$$

⁸ $\mathbf{X}(\mathbf{\Lambda})$ is the matrix with entries $\mathbf{X}(\mathbf{\Lambda}^\mu_\nu) = \mathbf{X}^\alpha \partial/\partial\zeta^\alpha (\Lambda^\mu_\nu)$.

To accomplish our enterprise we choose at an arbitrary $p_o \in \gamma$ normal coordiante functions such that

$$e_0(p_o) = \frac{1}{\sqrt{g_{00}(p_o)}} \gamma_*|_{p_o} = \partial/\partial\zeta^0|_{p_o}$$

and recall that from Proposition 5 there exists $e_i(p_o)$, $i = 1, 2, 3$ that together with $e_0(p_o)$ satisfy $g(e_\mu, e_\nu)|_{p_o \in \gamma} = \eta_{\mu\nu}$. So, our task is simply reduced to find solutions for Eq.(72)⁹. Now, taking $\mathbf{X} = \partial/\partial\zeta^\alpha$ we have that $(\mathbf{\Gamma}_\mathbf{X})^\mu_\nu := \mathbf{\Gamma}^{\mu\cdots}_{\cdot\alpha\nu}$ and Eq.(72) is the system of differential equations

$$\partial/\partial\zeta^\alpha (\Lambda^\mu_\kappa) = -\mathbf{\Gamma}^{\mu\cdots}_{\cdot\alpha\nu} \Lambda^\nu_\kappa \quad (75)$$

whose solution with given boundary conditions is well known [9]. Once that solution is known we have that $D_{e_\nu} e_\mu|_{p_o} = 0$ and thus we construct the **IMF** γ by simply parallel transporting the basis $\{e_\mu(p_o)\}$ of $T_{p_o}M$ along γ , getting for any $p \in \gamma$, $D_{e_\nu} e_\mu|_{p \in \gamma} = 0$. ■

Taking into account the previous proposition we finally propose the following:

Definition 31 *Given a timelike autoparallel line $\gamma \subset U \subset M$ in a Riemann-Cartan spacetime structure and coordinate functions $\langle\zeta^\mu\rangle$ covering $U \subset M$ we say that a reference frame $\mathbf{L} \in \text{sec}TU$ is a local inertial reference frame associated to γ (**LIRF** γ) iff for all $p \in \gamma$ there exists exists a section s of the orthogonal frame bundle $FU \subset FM, U \subset M$,*

$$s_\gamma = \{(e_0(p), e_1(p), e_2(p), e_3(p)), p \in U \subset M\} \subset s, \quad (76)$$

where

$$\begin{aligned} e_0|_{p \in \gamma} &= \mathbf{L}|_{p \in \gamma} = \frac{\partial}{\partial\zeta^0}\Big|_{p \in \gamma} = \gamma_*|_{p \in \gamma}, \\ \omega|_{p \in \gamma} &= -\mathbf{T}^0\Big|_{p \in \gamma}, \\ \mathbf{\Gamma}^{\nu\cdots}_{\cdot\mu\alpha}\Big|_{p \in \gamma} &= 0, \end{aligned} \quad (77)$$

Moreover we say that $\langle\zeta^\mu\rangle$ are inertial coordinates.

Remark 32 *Differently from the case of the **LIRF** in a Lorentzian spacetime, in a general Riemann-Cartan spacetime we do not have $g(\partial/\partial\zeta^\mu, \partial/\partial\zeta^\nu)|_{p \in \gamma} = \eta_{\mu\nu}$ and $\partial g_{\alpha\beta}/\partial\zeta^\mu|_{p \in \gamma} = 0$, for otherwise we get $\mathbf{\hat{\Gamma}}^{\nu\cdots}_{\cdot\mu\alpha}\Big|_{p \in \gamma} = 0$ which, as we saw above, implies a completely antisymmetric torsion if we want $\mathbf{\Gamma}^{\nu\cdots}_{\cdot\mu\alpha}(p)\Big|_{p \in \gamma} = 0$. Moreover, we observe that in the basis $\{e_\mu\}$ the components of the torsion tensor are according to Eq.(22) $\mathbf{T}^{\alpha\cdots}_{\cdot\mu\nu}|_{p \in \gamma} = -\mathbf{c}^{\alpha\cdots}_{\cdot\mu\nu}|_{p \in \gamma}$ and the components of*

⁹If these solutions result in a set of non orthonormal frames we get from them an orthonormal frame by standard procedures.

the Riemann curvature tensor are $\mathbf{R}_{\mu\alpha\beta}^{\lambda\cdot\cdot}\Big|_{p\in\gamma} = \mathbf{e}_\alpha(\gamma_{\beta\mu}^{\lambda\cdot\cdot})\Big|_{p\in\gamma} - \mathbf{e}_\beta(\gamma_{\alpha\mu}^{\lambda\cdot\cdot})\Big|_{p\in\gamma}$. Keep in mind that although $\gamma_{\alpha\mu}^{\lambda\cdot\cdot}|_\gamma = 0$, we have that the Riemann curvature tensor is non null in all $p \in \gamma$ since $\mathbf{e}_\alpha(\gamma_{\beta\mu}^{\lambda\cdot\cdot})\Big|_{p\in\gamma} \neq 0$.

5 Equivalence Principle and Einstein's Most Happy Thought

At last we want to comment that, as well known, in Einstein's GR one can easily distinguish (despite some claims on the contrary) in any *real physical laboratory*, (i.e., not one modelled by a timelike worldline) a true gravitational field from an acceleration field of a given reference frame in Minkowski spacetime [20, 17]. This is because in GR the *mark* of a real gravitational field is the non null Riemann curvature tensor of $\overset{e}{D}$, and the Riemann curvature tensor of $\overset{m}{D}$ (present in the definition of Minkowski spacetime) is null. However if we interpret a gravitational field as the torsion 2-forms of an *effective* teleparallel spacetime¹⁰ $(M, \boldsymbol{\eta}, \overset{e}{\nabla}, \boldsymbol{\tau}_\eta, \uparrow)$ viewed according to the ideas developed in [6, 19] as *deformation* of Minkowski spacetime, then one can also interpret the acceleration field of an accelerated reference frame in Minkowski spacetime as generating an effective teleparallel spacetime $(M, \boldsymbol{\eta}, \overset{e}{\nabla}, \boldsymbol{\tau}_\eta, \uparrow)$ structure. This can be done as follows. Let $\mathbf{Z} \in \text{sec } TU$, $U \subset M$ with $\boldsymbol{\eta}(\mathbf{Z}, \mathbf{Z}) = 1$ an *accelerated reference frame* on Minkowski spacetime. This means as we know from Section 3.1 that $\mathbf{a} = \overset{m}{D}_{\mathbf{Z}}\mathbf{Z} \neq 0$. Put $\mathbf{e}_0 = \mathbf{Z}$ and define an accelerated reference frame as *non* trivial if $\boldsymbol{\theta}^0 = \boldsymbol{\eta}(\mathbf{e}_0, \cdot)$ is not an exact differential. Next recall that in $V \subset M$ there always exist three other $\boldsymbol{\eta}$ -orthonormal vector fields \mathbf{e}_i , $i = 1, 2, 3$ such that $\langle \mathbf{e}_\mu \rangle$ is an $\boldsymbol{\eta}$ -orthonormal basis for TU , i.e., $\boldsymbol{\eta} = \eta_{\mu\nu} \boldsymbol{\theta}^\mu \otimes \boldsymbol{\theta}^\nu$, where $\langle \boldsymbol{\theta}^\mu \rangle$ is the dual basis¹¹ of $\langle \mathbf{e}_\mu \rangle$. We then have, $\overset{m}{D}_{\mathbf{e}_\alpha} \mathbf{e}_\beta = \gamma_{\alpha\beta}^{\kappa\cdot\cdot} \mathbf{e}_\kappa$, $\overset{m}{D}_{\mathbf{e}_\alpha} \boldsymbol{\theta}^\beta = -\gamma_{\alpha\kappa}^{\beta\cdot\cdot} \boldsymbol{\theta}^\kappa$.

What remains in order to be possible to interpret an acceleration field as a kind of 'gravitational field' is to introduce on M a $\boldsymbol{\eta}$ -metric compatible connection $\overset{e}{\nabla}$ such that the $\{\mathbf{e}_\mu\}$ is teleparallel according to it, i.e., $\overset{e}{\nabla}_{\mathbf{e}_\alpha} \mathbf{e}_\beta = 0$, $\overset{e}{\nabla}_{\mathbf{e}_\alpha} \boldsymbol{\theta}^\beta = 0$. Indeed, with this connection the structure $\langle M \simeq \mathbb{R}^4, \boldsymbol{\eta}, \overset{e}{\nabla}, \boldsymbol{\tau}_\eta, \uparrow \rangle$ has null Riemann curvature tensor but a non null torsion tensor, whose components are related¹² with the components of the acceleration \mathbf{a} and with the other coefficients $\gamma_{\alpha\kappa}^{\beta\cdot\cdot}$ of the connection $\overset{m}{D}$, which describe the motion on Minkowski spacetime of a *grid* represented by the orthonormal frame $\langle \mathbf{e}_\mu \rangle$. Schücking [23] thinks that such a description of the gravitational field makes Einstein most happy though, i.e., the equivalence principle (understood as equivalence between

¹⁰A teleparallel spacetime is one equipped with a metric compatible connections for which its Riemann curvature tensor is null, but its torsion tensor is non null.

¹¹In general we will also have that $d\boldsymbol{\theta}^i \neq 0$, $i = 1, 2, 3$.

¹²The explicit formulas can be easily derived using the equations of section 4.5.8 of [18] which generalizes for connections with non null *nonmetricity* tensors Eq.(17).

acceleration and gravitational field) a legitimate mathematical idea. However, a *true* gravitational field must satisfy (at least with good approximation) Einstein equation or the *equivalent* equation for the tetrad fields $\langle \mathbf{e}_\mu \rangle$ [6, 19], whereas there is no single reason for an acceleration field to satisfy that equation.

6 Conclusions

In this paper we have recalled the definitions of observers, reference frames and naturally adapted coordinate chart to a given reference frame. Equipped with these definitions and some basic results such as the proper meaning of an inertial reference frame in Minkowski spacetime and the notion of pseudo-inertial reference frames and locally inertial reference frames in a Lorentzian spacetime, we showed how to define consistently *locally inertial reference systems* in a general Riemann-Cartan spacetime structure $\langle M, \mathbf{g}, D, \boldsymbol{\tau}_{\mathbf{g}}, \uparrow \rangle$. We proved that a set of normal coordinate functions $\langle \zeta^\mu \rangle$ covering a timelike autoparallel do not automatically define a **LIRF** in $\langle M, \mathbf{g}, D, \boldsymbol{\tau}_{\mathbf{g}}, \uparrow \rangle$ as it is the case in a Lorentzian spacetime (recall section 4.2), but the coordinate basis $\langle \partial/\partial\zeta^\mu \rangle$ associated to the normal coordinate functions $\langle \zeta^\mu \rangle$ can be used to define a **LIRF** (Definition 31) once we take into account Proposition 30. We also briefly recalled how the concepts of **LIRF** in Lorentzian and Riemann-Cartan spacetimes are linked to “Einstein’s most happy thought”, i.e., the equivalence principle. Summing up we think that our paper complement and help to clarify presentations of related issues appearing in excellent papers [1, 7, 8, 9, 10, 11, 14, 24], besides clarifying some misconceptions like the ones in [4, 5, 12, 21, 23] as exposed above.

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